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Vacuum energy in smooth background fields

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Abstract. We consider the ground-state energy of a scalar field in the background of a general potential which depends on one coordinate. We consider a general expression following from the analytical properties of the one-dimensional scattering matrix. We show that reflections give a positive and bound states a negative contribution to the ground-state energy and we calculate explicitly two simple examples, the square-well potential and a piecewise oscillatory potential. We demonstrate our formulae by an easy rederivation of the mass of the kink.

1. Introduction

The ground-state energy is the response of the vacuum to some external conditions or background fields. The most popular example is the Casimir effect. Important applications arise for spaces with non-trivial topologies, especially in connection with spontaneous compactification, within external gravitational fields, in the bag model and other areas. Sharp boundary conditions are often a good approximation of a physical situation. However, there is an interest in calculating the ground-state energy in cases with smooth background fields, too. Possible applications include non-ideal boundaries, general background fields and metrics and others. Special interest comes from classical solutions of field equations like solitons, which can be considered as smooth background fields when calculating quantum fluctuations around them.

The calculation of the ground-state energy is quite easy and powerful methods—the zeta function method [1] and the heat kernel expansion, for instance—are known for a variety of different boundary conditions and topologies, and also in constant background fields. There is a full understanding of the renormalization. However, the calculation of the bound-state energy in general background fields is still a difficult task. In the general case one is left with perturbative expansions with respect to the background field or with respect to its derivatives.

The aim of the present paper is to make a step forward in the investigation of the ground-state energy in a general background field $\Phi(x)$. For this reason we consider the effective potential†

$$V_{\mathrm{eff}} = \tfrac{1}{2}\mathrm{Tr}\log\left(\Box - m^2 - V(x)\right) \tag{1}$$

in the external potential $V(x) \equiv \lambda'\Phi(x)^2$. Then the problem is to find the spectrum of this operator. For a time-independent potential $V(x)$ this is equivalent to a non-relativistic Schrödinger equation, which is a well investigated object. Therefore, it is natural to adopt

† Usually this term is used in the case of a constant background field. It results in the one-loop contribution to the effective action $\Gamma[\Phi] = \int \mathrm{d}x\ (-V_{\mathrm{eff}})$.

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these results. There are several attempts to do so in literature. For instance this idea has been proposed in [2] and recently in [3]. Also, the expression of the effective potential as a frequency sum over the mode density or the scattering phase shift is commonly used (see [4] or recently [6]). Actual interest is in connection with chiral fields [5] and the evolution of bubble walls [7].

In the present paper we restrict ourself to the case when $V(x)$ depends on one coordinate only and decreases as $x \to \infty$ so that $\int_{-\infty}^{\infty} |V(x)|(1+|x|)\, dx$ is finite. This case is the simplest one. It is generic in $(1+1)$ dimensions and it serves as a model in $(3+1)$ dimensions. Using well known properties of the S-matrix of the corresponding non-relativistic Schrödinger equation we give a closed representation for the effective potential. By means of an analytic continuation we obtain a representation which incorporates the bound states in a very natural manner and which has an improved convergence. Using this representation we consider two simple examples—the square-well potential and a piecewise quadratic potential—and calculate the ground-state energy numerically. A different representation is obtained in using the analytical properties which allow for a general conclusion concerning the sign of the effective potential. An explicit calculation is possible for all reflectionless potentials. We give the corresponding formulae. We apply them to the kink model in $(1+1)$ dimensions and rederive the corresponding result [11].

The case of a potential depending on one coordinate which is considered here, is of restricted interest. The next step will be the extension of this method to a spherical symmetric potential, for instance.

2. *Mode summation and analytic continuation*

For its renormalization the effective potential requires imbedding into an external system. In order to have a concrete model we consider the Lagrange density

$$L = \tfrac{1}{2}\Phi\left(\Box - M^2 - \lambda\Phi^2\right)\Phi + \tfrac{1}{2}\varphi\left(\Box - m^2 - \lambda'\Phi^2\right)\varphi \,. \tag{2}$$

It can be thought of as the result of the expansion of the action around a classical solution. The field Φ should be considered as a classical background field depending on one coordinate. By means of

$$V(x_1) = \lambda'\Phi^2 \tag{3}$$

it defines the potential in (1) for the field $\varphi(x)$, which should be quantized in the usual way. The energy density per unit area of the plane perpendicular to x_1 is given by

$$E = \tfrac{1}{2}V_g + \tfrac{1}{2}M^2 V_1 + \lambda V_2 + V_{\text{eff}} \tag{4}$$

with the following notation: $V_g \equiv \int_{-\infty}^{\infty} dx_1\, ((\partial/\partial x_1)\Phi(x_1))^2$, $V_1 \equiv \int_{-\infty}^{\infty} dx_1\, (\Phi(x_1))^2$ and $V_2 \equiv \int_{-\infty}^{\infty} dx_1\, (\Phi(x_1))^4$. The effective potential V_{eff} is given by (1), and a regularization is provided. We use the regularization known from the zeta function method

$$V_{\text{eff}} = \frac{1}{2}\frac{\partial}{\partial s}\text{Tr}\left(\Box - m^2 - V(x_1)\right)^{-s}\mu^{2s}\Big|_{s=0} \,. \tag{5}$$

The parameter μ has the dimension of a mass and appears in order to adjust the dimensions.

Instead of the effective potential one can use the ground-state energy (for a detailed discussion see [12])

$$E_0 = \frac{1}{2}\int \frac{d^2k}{(2\pi)^2}\sum_n \sqrt{k_1^2 + k_2^2 + \omega_n^2 + m^2}^{\,(1-2s)}\mu^{2s} \tag{6}$$

with s to zero in the end.

The system has to be considered within a large box $-L \leqslant x_1 \leqslant L$ because E_0 as well as $V_{\rm eff}$ contain a contribution which is proportional to L. It appears as a consequence of the translational invariance which is unbroken in the case when the potential $V(x)$ is absent. Therefore, that contribution is independent on $V(x)$ and can be dropped. The corresponding Schrödinger equation reads

$$\left(-\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = k^2\,\psi(x)\,. \tag{7}$$

For $-L \leqslant x_1 \leqslant L$, the eigenvalues $k \to \omega_n$ are discrete.

Because the potential depends on one coordinate only, after integration over the momenta corresponding to the directions with unbroken translational invariance, the effective potential can be written in the form

$$V_{\rm eff} = -\frac{1}{2}\frac{\partial}{\partial s}\frac{\Gamma\left(s-\frac{3}{2}\right)}{8\pi^{3/2}\Gamma(s)}\sum_n \left(\omega_n^2 + m^2\right)^{3/2-s}\Big|_{s=0}\,. \tag{8}$$

The frequency sum in this expression is the essential quantity to be calculated. It is known to have a single pole at $s = 0$:

$$\sum_n \left(\omega_n^2 + m^2\right)^{3/2-s} = \frac{1}{s}\Sigma_{-1} + \Sigma_0 + \mathrm{O}(s) \tag{9}$$

where Σ_{-1} is the residuum and Σ_0 is the regular part.

Using this, the effective potential takes the form

$$V_{\rm eff} = -\frac{\Sigma_{-1}}{6\pi}\left(\tfrac{4}{3} + \ln(\tfrac{1}{2}\mu)\right) - \frac{\Sigma_0}{12\pi}\,. \tag{10}$$

A similar calculation yields for the ground-state energy

$$E_0 = -\frac{1}{s}\frac{\Sigma_{-1}}{12\pi} - \frac{\Sigma_{-1}}{6\pi}\left(\tfrac{1}{3} + \ln\mu\right) - \frac{\Sigma_0}{12\pi} + \mathrm{O}(s)\,. \tag{11}$$

The renormalization will be discussed later on. In order to calculate the frequency sum (9) we use the following well known properties of the one-dimensional Schrödinger equation. Under the asymption of a sufficiently quickly decreasing potential $V(x)$ the two independent solutions can be chosen to have the asymptotics

$$\begin{aligned} &\psi_1 \underset{x\to-\infty}{\simeq} \mathrm{e}^{ikx} + s_{12}\,\mathrm{e}^{-ikx} && \psi_1 \underset{x\to\infty}{\simeq} s_{11}\,\mathrm{e}^{ikx} \\ &\psi_2 \underset{x\to-\infty}{\simeq} s_{22}\,\mathrm{e}^{-ikx} && \psi_2 \underset{x\to\infty}{\simeq} s_{21}\,\mathrm{e}^{ikx} + \mathrm{e}^{-ikx}\,. \end{aligned} \tag{12}$$

The **S**-matrix $\mathbf{S} = (s_{ij})$ is unitary. The coefficient $s_{11}(k)$ is a meromorphic function of k with possibly a finite number of simple poles on the upper half of the imaginary axis at $k = i\kappa$, where κ_n are the corresponding bound-state energies. The reflection and the transmission coefficients are given by $T(k) = |s_{11}(k)|^2$ and $R(k) = |s_{12}(k)|^2$, respectively. Obviously, $1 = R + T$ holds.

At $x = \pm L$ we impose Dirichlet boundary conditions on $\psi(x)$. Other conditions could be chosen as well, the difference between distinct choices can be shown to be independent for the potential.

Under these conditions, the eigenvalues ω_n are solutions of the equations

$$(s_{11} \pm s_{21})\,\mathrm{e}^{ikL} \pm \mathrm{e}^{-ikL} = 0\,.$$

These conditions correspond to the linear combinations $\psi_1 \pm \psi_2$, which also form an independent set of solutions. Now, we express the sum (9) in the form

$$\sum_{\omega_n} \left(\omega_n^2 + m^2\right)^{3/2-s} = \sum_{\kappa_n} \left(-\kappa_n^2 + m^2\right)^{3/2-s} + \int_\gamma \frac{\mathrm{d}k}{2\pi \mathrm{i}} \left(k^2 + m^2\right)^{3/2-s} \frac{\partial}{\partial k} \ln\left[\left((s_{11} - s_{21})\, \mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right) \times \left((s_{11} + s_{21})\, \mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] \tag{13}$$

where the integration path γ goes from ∞ above the real axis to 0 and further below the real axis from 0 to ∞ so that the points $k = \omega_n > 0$ (which are poles of the integrand) are inside. In the right-hand side of this formula, the sum corresponds to the bound states $k = \mathrm{i}\kappa_n$ and the integral corresponds to the contribution of the continuous part of the spectrum in the limit $L \to \infty$.

To perform the limit $L \to \infty$ we consider the upper half part of γ. By means of $k \to k + \mathrm{i}\epsilon$ ($\epsilon > 0$) we have

$$\ln\left[\left((s_{11} - s_{21})\, \mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right)\left((s_{11} + s_{21})\, \mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] = -2\mathrm{i}kL + 2\epsilon L - \mathrm{i}\pi + \mathrm{O}\left(\mathrm{e}^{-\epsilon L}\right).$$

For the lower half part we note $k \to k - \mathrm{i}\epsilon$ and have

$$\ln\left[\left((s_{11} - s_{21})\, \mathrm{e}^{\mathrm{i}kL} - \mathrm{e}^{-\mathrm{i}kL}\right)\left((s_{11} + s_{21})\, \mathrm{e}^{\mathrm{i}kL} + \mathrm{e}^{-\mathrm{i}kL}\right)\right] = 2\mathrm{i}kL + 2\epsilon L + \ln\left(s_{11}^2 - s_{21}^2\right) + \mathrm{O}\left(\mathrm{e}^{-\epsilon L}\right).$$

In the limit $L \to \infty$ we obtain up to exponentially small contributions

$$\sum_{\omega_n} \left(\omega_n^2 + m^2\right)^{3/2-s} = \sum_{\kappa_n} \left(-\kappa_n^2 + m^2\right)^{3/2-s} + 2L \int_0^\infty \frac{\mathrm{d}k}{\pi} \left(k^2 + m^2\right)^{3/2-s} + \int_0^\infty \frac{\mathrm{d}k}{2\pi \mathrm{i}} \left(k^2 + m^2\right)^{3/2-s} \frac{\partial}{\partial k} \ln\left(s_{11}^2 - s_{21}^2\right) + \mathrm{O}\left(\mathrm{e}^{-\epsilon L}\right). \tag{14}$$

The second term in the right-hand side is the contribution, which is independent on $V(x)$ and which is proportional to L. It will be dropped. Now, the integration path can be turned to the imaginary axis: $k \to \mathrm{i}k$. Using the relation

$$s_{11}^2 - s_{21}^2 = \frac{s_{11}(k)}{s_{11}(-k)} = \mathrm{e}^{2\mathrm{i}\delta(k)}$$

which is a consequence of the unitarity of $\mathbf{S}$ and where $\delta(k)$ is the scattering phase, we obtain

$$\sum_{\omega_n}{}' \left(\omega_n^2 + m^2\right)^{3/2-s} = -\frac{\cos \pi s}{\pi} \int_m^\infty \mathrm{d}k \left(k^2 - m^2\right)^{3/2-s} \frac{\partial}{\partial k} \ln s_{11}(\mathrm{i}k) \tag{15}$$

up to terms independent of $V(x)$. The contribution from the bound states has been cancelled by the extra contributions appearing from the poles at $k = \mathrm{i}\kappa_n$ when turning the integration contour to the imaginary axis. This formula connects the ground-state energy with the $\mathbf{S}$-matrix taken at imaginary momentum k. It is remarkable that the contributions of the bound states are not here explicitly. They are implicitly present, of course.

Now, let us discuss the renormalization. As was shown in [8], the function $\log s_{11}$ has the following asymptotics as $k \to \infty$:

$$\log s_{11}(\mathrm{i}k) = -\frac{\lambda' V_1}{2k} + \frac{\lambda'^2 V_2}{(2k)^3} + \mathrm{O}\left(k^{-5}\right)$$

with V_1 and V_2 defined below (4). Inserting these two terms into the representation (15) of the frequency sum, we can calculate the resulting residuum at $s = 0$ and the regular part according to (9). We denote them by Σ_{-1}^{as} and Σ_0^{as}, respectively. Now, the effective potential V_{eff} splits into two parts $V_{eff} = V_{eff}^{as} + V_{eff}^{sub}$ with

$$V_{eff}^{as} = \frac{\lambda'}{32\pi^2}\left(\log\frac{m^2}{\mu^2} - 1\right)m^2 V_1 + \frac{\lambda'^2}{64\pi^2}\log\frac{m^2}{\mu^2}V_2 . \tag{16}$$

Inserting the last line into (4) for the energy of the whole system and using (3) it is clear that these terms perform a renormalization of the mass M and the coupling λ of the field Φ:

$$M^2 \to M^2 + \frac{m^2\lambda'}{16\pi^2}\left(\log\frac{m^2}{\mu^2} - 1\right) \qquad \lambda \to \lambda + \frac{\lambda'^2}{64\pi^2}\log\frac{m^2}{\mu^2} . \tag{17}$$

This renormalization is a finite one. The reason is the use of the zeta function, which yields finite results. When using (6) for the ground-state energy instead, the same calculation can be performed. In that case, the renormalization reads

$$M^2 \to M^2 + \frac{m^2\lambda'}{16\pi^2}\left(\log\frac{m^2}{4\mu^2} + 1 - \frac{1}{s}\right) \qquad \lambda \to \lambda + \frac{\lambda'^2}{64\pi^2}\left(\log\frac{m^2}{4\mu^2} + \frac{1}{2} - \frac{1}{s}\right) . \tag{18}$$

It is infinite, as is usual in quantum field theory. Also, from the general theory of renormalization, these coefficients can be connected with the renormalization group functions—especially with the beta function, of course. Notice that in this case there is no renormalization of the kinetic term, i.e. of that term which contains derivatives of the field Φ.

Substracting the asymptotic terms (16) from $\log s_{11}(ik)$ in (15), the integral becomes finite for $s = 0$. The corresponding contribution to the effective potential reads

$$V_{eff}^{sub} = \frac{1}{12\pi^2}\int_m^\infty dk\,\left(k^2 - m^2\right)^{3/2}\frac{\partial}{\partial k}\left[\log s_{11}(ik) + \frac{\lambda' V_1}{2k} - \frac{\lambda'^2 V_2}{8k^3}\right] . \tag{19}$$

3. Two simple examples

In this section we apply the above formulae to two simple examples for the potential $V(x)$, namely the square-well potential†

$$V_{sq}(x) = \begin{cases} 0 & |x| > L \\ V_0 & |x| < L \end{cases} \tag{20}$$

and the piecewise oscillatory potential

$$V_{os}(x) = \begin{cases} 0 & |x| > L \\ V_0(1 - |x|/L)^2 & |x| \leqslant L . \end{cases} \tag{21}$$

Let us first consider the square-well potential V_{sq} (20). The contribution V_g to the energy, which results from the gradient of the field Φ, is infinite; the other two parts are simply $V_1 = LV_0$ and $V_2 = LV_0^2$. Therefore, the total energy is not a meaningful quantity, although the calculated effective potential and the renormalization can be carried out.

† In this section L denotes the (finite) size of the potential $V(x)$, not to be confused with L in section 2 which was used as a regulator.

The **S**-matrix for this potential is well known and it reads (already after the rotation to the imaginary axis $k \to \mathrm{i}k$)

$$s_{11}(\mathrm{i}k) = \frac{4kq\,\mathrm{e}^{2kL}}{(k+q)^2\,\mathrm{e}^{2qL} - (k-q)^2\,\mathrm{e}^{-2qL}} \tag{22}$$

with $q = \sqrt{k^2+V_0}$. This expression can be inserted into $V_{\mathrm{eff}}^{\mathrm{sub}}$ (19). After some simple calculations we obtain

$$\begin{aligned} V_{\mathrm{eff}}^{\mathrm{sub}} = {} & \frac{L}{32\pi^2}\left(\left(m^2+V_0\right)^2\log\left(1+\frac{V_0}{m^2}\right) - m^2V_0 - \tfrac{3}{2}V_0^2\right) \\ & + \frac{1}{12\pi^2}\int_m^\infty \mathrm{d}k\,\left(k^2-m^2\right)^{3/2}\,\frac{(k-q)^2}{kq^2} \\ & - \frac{1}{3\pi^2}\int_m^\infty \mathrm{d}k\,\left(k^2-m^2\right)^{3/2}\,\frac{1}{q}\,\frac{1+kL}{\left(\frac{k+q}{k-q}\right)^2\mathrm{e}^{4qL}-1}\,. \end{aligned} \tag{23}$$

It is represented in figure 1 for several values of the parameters L and V_0. The first term within this expression is proportional to L, the 'length of the potential well'. Its sign depends on V_0/m^2 and takes both values, positive and negative. The second part depends on V_0/m^2 only. The third part depends on all parameters. For large L it is proportional to $\exp(-4\sqrt{m^2+V_0}L)$. The convergence of the k-integral is a consequence of the subtractions performed. In the second term the convergence of the integral is by powers of k. In the third term, the most complicated, the integrand falls off exponentially as $k \to \infty$ and the integral converges very easily. This is a result of rotating the integration contour in V_{eff} (15).

The representation (23) is valid for both signs of V_0, i.e. for repulsive as well as for attractive potentials $V(x_1)$. In the last case, there is always at least one bound state. Its contribution to the frequency sum is taken into account by (19) automatically. In

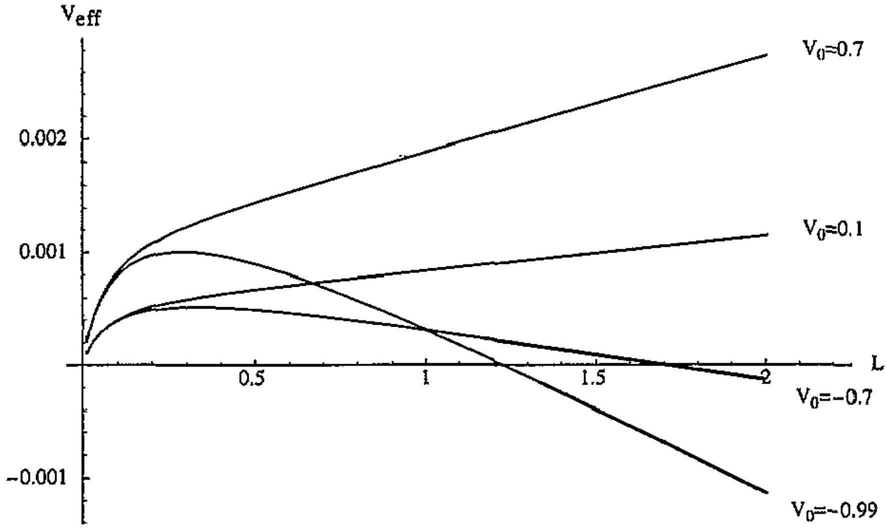

Figure 1. The effective potential for the square-well potential as function of the length L of the well for some values of its height V_0.

the case $V_0 < -m < 0$, the potential is over-critical with respect to particle creation. Correspondingly, $V_{\rm eff}^{\rm sub}$ acquires an imaginary part.

Now we turn to the piecewise oscillatory potential $V_{\rm os}$ (21). The scattering matrix can be obtained in the following way. We consider the solution of the Schrödinger equation (7)

$$\psi(x) = \begin{cases} e^{ikx} + s_{12}e^{-ikx} & x < -L \\ \alpha_1 u(-x) + \beta_1 v(-x) & -L \leqslant x \leqslant 0 \\ \alpha_2 u(x) + \beta_2 v(x) & 0 \leqslant x \leqslant L \\ s_{11}e^{ikx} & L < x \end{cases}$$

where $u(x)$ and $v(x)$ are two independent solutions in the interval $0 \leqslant x \leqslant L$, and α and β are constants. This function and its first derivative have to be continuous at $x = 0$ and $x = \pm L$. From these conditions, one obtains

$$s_{11}(k) = \frac{ik\, W^2\, e^{-2ik}}{(Q\, u(0) - P\, v(0))\,(Q\, u'(0) - P\, v'(0))} \tag{24}$$

with $W = uv' - u'v$ and where the abbreviations $P = u'(L) - iku(L)$ and $Q = v'(L) - ik\, v(L)$ are used. These solutions can be expressed by hypergeometric functions. We choose the following combinations [9]:

$$u(x) = U(a, y) \tag{25}$$

$$v(x) = U(a, -y) \tag{26}$$

with the notation

$$a \equiv \frac{-k^2 L}{2\sqrt{V_0}} \qquad y \equiv \sqrt{\frac{2\sqrt{V_0}}{L}}\,(L - x)$$

and

$$U(a, y) = \frac{\sqrt{\pi}}{2^\alpha}\left[\frac{{}_1F_1(\alpha, 1/2; y^2/2)}{\Gamma(\alpha + 1/2)} - \frac{\sqrt{2}y\ {}_1F_1(\alpha + 1/2, 3/2; y^2/2)}{\Gamma(\alpha)}\right] e^{-y^2/4}$$

with $\alpha \equiv \frac{1}{2}a + \frac{1}{4}$. From [9] we find $W = -\sqrt{4\pi\sqrt{V_0}/L}/\,\Gamma(a + \frac{1}{2})$ in this case. Using these formulae, we obtain

$$Q\, u(0) - P\, v(0) = -\sqrt{\frac{4\pi\sqrt{V_0}}{L}}\left[{}_1F_1\left(\alpha, \tfrac{1}{2}; \tfrac{1}{2}\bar{L}^2\right) + \sqrt{a}\bar{L}\,{}_1F_1\left(\alpha + \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{1}{2}\bar{L}^2\right)\right] e^{-\bar{L}^2/4}$$

and

$$\begin{aligned} Q\, u'(0) - P\, v'(0) \equiv \Bigg\{ & 2\sqrt{\frac{2\pi}{L}}V_0^{3/4}\left(-\tfrac{1}{2}\,{}_1F_1\left(\alpha, \tfrac{1}{2}; \tfrac{1}{2}\bar{L}^2\right) + {}_1F_1'\left(\alpha, \tfrac{1}{2}; \tfrac{1}{4}\bar{L}^2\right)\right) \\ & + \sqrt{2\pi a}\frac{2\sqrt{V_0}}{L}\left(\left(1 - \tfrac{1}{2}\bar{L}^2\right){}_1F_1\left(\alpha + \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{1}{2}\bar{L}^2\right)\right. \\ & \left. + \bar{L}\,{}_1F_1'\left(\alpha + \tfrac{1}{2}, \tfrac{3}{2}; \tfrac{1}{2}\bar{L}^2\right)\right)\Bigg\} e^{-\bar{L}^2/4} \end{aligned}$$

with $\bar{L} \equiv \sqrt{2L\sqrt{V_0}}$ and ${}_1F'_1(a, b; z)$ being the derivative of the hypergeometric function with respect to z. These formulae are sufficient to calculate $s_{11}(k)$, and the rotation $k \to ik$ can also be performed. Before carrying out the calculation of the effective potential the subtractions have to be performed. From equation (21) we calculate $V_1 = \frac{2}{3}V_0 L$ and $V_2 = \frac{2}{5}V_0^2 L$. Inserting all of these into the effective potential $V_{\rm eff}^{\rm sub}$ (equation (19)), the

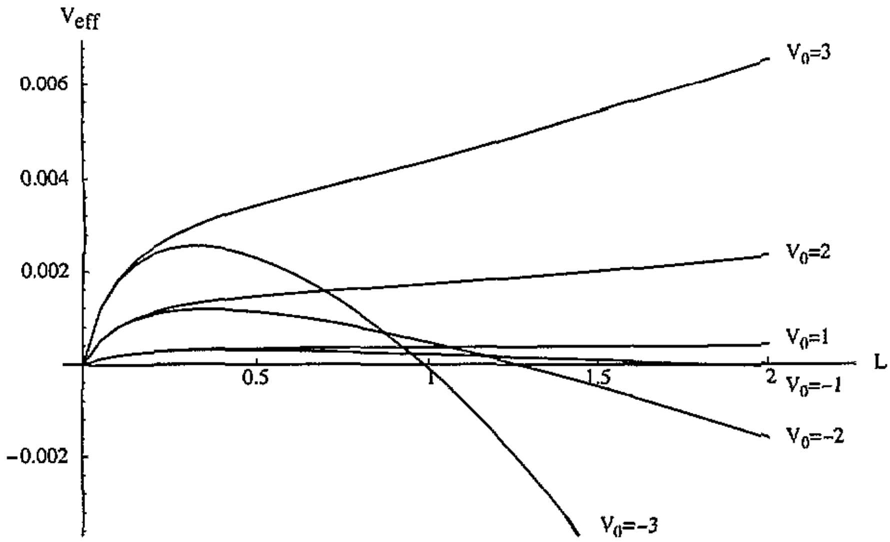

Figure 2. The effective potential for the piecewise oscillatory potential V_{os} as function of the lenght L for some values of the height V_0 for $m = 1$.

integral is finite and can be calculated numerically for any values of the parameters. The result is represented in figure 2 as a function of L for several values of the depth V_0. For negative V_0, the potential is attractive. In that case one has to perform the formal substitution $a \to -\mathrm{i}a$ and $\bar{L} \to \sqrt{\mathrm{i}}\bar{L}$ in the above formulae. For sufficiently large values of V_0, the potential becomes over-critical and the effective potential takes complex values.

4. Analytical properties and reflectionless potentials

Consider the Schrödinger equation (7). The set $\{R(k), \beta_n, \kappa_n (n = 0, 1, \dots, N)\}$, where $R(k)$ is the reflection coefficient, the κ_n are the bound-state energies, the β_n are some numbers and N is the number of bound states, is called scattering data. The potential $V(x)$ can be restored from the scattering data uniquely. In general, by means of its analytic properties, the coefficient $s_{11}(k)$ can be represented in the form

$$\log s_{11}(k) = \frac{-1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\log\left(1 - |R(q)|^2\right)}{k - q + \mathrm{i}\epsilon}\, \mathrm{d}q + \sum_{n=1}^{N} \log \frac{k + \mathrm{i}\kappa_n}{k - \mathrm{i}\kappa_n} \tag{27}$$

where $\epsilon \to +0$ and $R(q) = 1 - |s_{11}(q)|^2$ is the reflection coefficient and the sum goes over the bound states. The continuation to the positive imaginary axis, i.e. $k \to \mathrm{i}k$ yields

$$\log s_{11}(\mathrm{i}k) = \frac{k}{\pi} \int_{0}^{\infty} \frac{\log |s_{11}(q)|}{q^2 + k^2}\, \mathrm{d}q + \sum_{n=1}^{N} \log \frac{k + \kappa_n}{k - \kappa_n}$$

where $s_{11}(-k) = \overline{s_{11}(k)}$ has been used. In order to perform the renormalization we use the formulae [8]

$$\lambda' V_1 \equiv \int_{-\infty}^{\infty} V(x)\,\mathrm{d}x = \frac{-2}{\pi} \int_{-\infty}^{\infty} \log |s_{11}(q)|\,\mathrm{d}q - 4\sum_{n=1}^{N} \kappa_n$$

$$\lambda'^2 V_2 \equiv \int_{-\infty}^{\infty} V(x)^2\,\mathrm{d}x = \frac{-8}{\pi} \int_{-\infty}^{\infty} q^2 \log |s_{11}(q)|\,\mathrm{d}q + \tfrac{16}{3}\sum_{n=1}^{N} \kappa_n^3$$

and obtain from (19)

$$V_{\mathrm{eff}}^{\mathrm{sub}} = \frac{1}{12\pi^2} \int_m^{\infty} \mathrm{d}k (k^2 - m^2)^{3/2} \frac{\partial}{\partial k} \left\{ \frac{2}{\pi} \int_0^{\infty} \mathrm{d}q \frac{q^4}{k^2(k^2+q^2)} \log |s_{11}(q)| + \sum_{n=1}^{N} \left(\log \frac{k+\kappa_n}{k-\kappa_n} - 2\frac{\kappa_n}{k} + \frac{2}{3}\left(\frac{\kappa_n}{k}\right)^3 \right) \right\}. \tag{28}$$

This formula can be simplified:

$$V_{\mathrm{eff}}^{\mathrm{sub}} = \frac{1}{8\pi^3} \int_0^{\infty} \mathrm{d}q\, q \left\{ -2q + \sqrt{q^2+m^2} \log \frac{\sqrt{q^2+m^2}+q}{\sqrt{q^2+m^2}-q} \right\} \log \frac{1}{|s_{11}(q)|} - \frac{1}{6\pi^2} \sum_{n=1}^{N} \left[\left(m^2 - \kappa_n^2\right)^{3/2} \sin^{-1} \frac{\kappa_n}{m} + \tfrac{4}{3}\kappa_n^3 - \kappa_n m^2 \right]. \tag{29}$$

From this formula it follows that the bound states give a negative contribution to the effective potential whereas the integral in the right-hand side of this formula, which results from the reflections (it vanishes for $|s_{11}| = 1$, i.e. for a vanishing reflection coefficient), gives a positive contribution (note $|s_{11}| \leqslant 1$ and the expressions in both figure brackets in (29) are non-negative). This conclusion is different from what one would expect from the known fact, that a repulsive (resp. attractive) potential yields a negative (resp. positive) phase shift.

An important special case are the reflectionless potentials. In that case we have $|s_{11}(k)| = 1$ and $s_{11}(k)$ is a rational function

$$s_{11}(k) = \prod_{n=1}^{N} \frac{k + \mathrm{i}\kappa_n}{k - \mathrm{i}\kappa_n}. \tag{30}$$

The potential $V(x)$ can be restored from the scattering data explicitly

$$V(x) = -2\frac{\mathrm{d}^2}{\mathrm{d}x^2} \log \det \mathbf{A} \tag{31}$$

where the matrix $\mathbf{A}$ is

$$\mathbf{A}_{nm} = \delta_{nm} + \frac{\beta_n}{\kappa_n + \kappa_m} \mathrm{e}^{-2\kappa_n x}.$$

The effective potential is in this case given by the second term in right-hand side of (29):

$$V_{\mathrm{eff}}^{\mathrm{sub}} = -\frac{1}{6\pi^2} \sum_{n=1}^{N} \left[\left(m^2 - \kappa_n^2\right)^{3/2} \sin^{-1} \frac{\kappa_n}{m} + \tfrac{4}{3}\kappa_n^3 - \kappa_n m^2 \right].$$

It is completely negative. The contribution of one bound state to this formula is shown in figure 3(a).

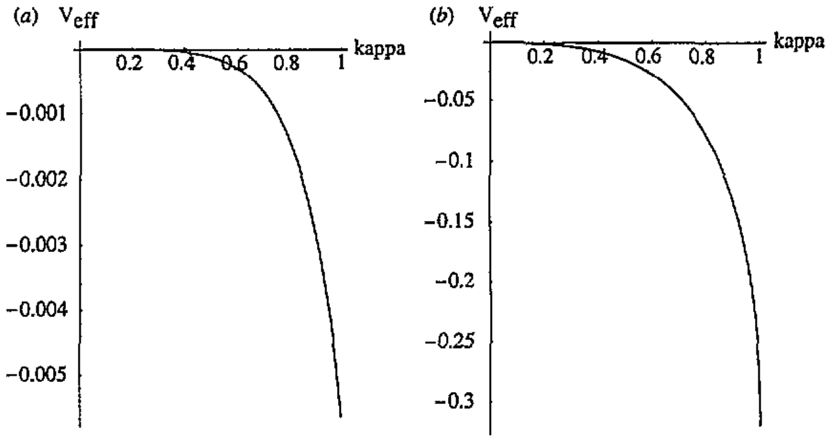

Figure 3. (*a*) The contribution of one bound state to the effective potential for a reflectionless potential in the (3+1)-dimensional case. (*b*) The contribution of one bound state to the effective potential for a reflectionless potential in the (1 + 1)-dimensional case.

5. The kink

In this section we demonstrate how our method reproduces the well known correction to the mass of the kink. For this purpose we need to rewrite the basic formulae in (1 + 1) dimensions. The formulae (2)–(5) remain unchanged, and instead of (8) we obtain

$$V_{\rm eff} = -\frac{1}{2}\frac{\partial}{\partial s}\frac{\Gamma(s-1/2)}{2\pi^{1/2}\Gamma(s)}\sum_n \left(\omega_n^2+m^2\right)^{1/2-s}\Big|_{s=0}.$$

Expression (9) is independent on the dimension and (10) becomes

$$V_{\rm eff} = \Sigma_{-1}\left(1+\ln\left(\tfrac{1}{2}\mu\right)\right) - \tfrac{1}{2}\Sigma_0 .$$

The renormalization has been changed in that sense, so that only one subtraction is necessary and only the mass term is renormalized. For the effective potential we obtain

$$V_{\rm eff}^{\rm sub} = \frac{1}{2\pi}\int_m^\infty {\rm d}k\,\left(k^2-m^2\right)^{1/2}\frac{\partial}{\partial k}\left[\log s_{11}({\rm i}k) + \frac{\lambda' V_1}{2k}\right].$$

In the case of a reflectionless potential we note (30) and this integral can be calculated. We obtain for the effective potential

$$\begin{aligned} V_{\rm eff}^{(i)} &= \sum_{n=1}^{N}\frac{1}{2\pi}\int_m^\infty {\rm d}k\,\left(k^2-m^2\right)^{1/2}\frac{\partial}{\partial k}\left[\log\frac{k+\kappa_n}{k-\kappa_i} - 2\frac{\kappa_n}{k}\right] \\ &= \sum_{n=1}^{N}\frac{1}{\pi}\left[\left(m^2-\kappa_n^2\right)^{1/2}\sin^{1}\frac{\kappa_n}{m} - \kappa_n\right]. \end{aligned} \qquad (32)$$

This function is very close to that in the (3+1)-dimensional case. It is shown in figure 3(*b*). In this form it was first derived in [10] in connection with static solitons.

These formulae can easily be applied to the kink. It is a static solution $\Phi_{\rm cl}$ of the scalar Φ^4 theory in the case of a mass term with 'wrong sign'

$$S(\Phi) = \frac{1}{2}\int {\rm d}x\left(\left(\frac{\partial\Phi}{\partial x^\mu}\right)^2 + M^2\Phi^2 - \tfrac{1}{2}\lambda\Phi^4\right).$$

It reads $\Phi_{\rm cl}(x) = (M/\sqrt{\lambda})\tanh(mx/\sqrt{2})$ By means of a expansion of the action around the classical solution $\Phi = \Phi_{\rm cl} + \varphi$, i.e.

$$S(\Phi_{\rm cl} + \varphi) = S(\Phi_{\rm cl}) + \frac{1}{2}\int {\rm d}x\ \varphi(x)\frac{\delta^2 S(\Phi_{\rm cl})}{\delta\varphi^2}\varphi(x) + \cdots$$

we obtain the Lagrangian (2) with $m = \sqrt{2}M$ and $V(x_1) = 3M^2/\cosh^2(mx_1/\sqrt{2})$. This is the well known Eckhart potential. It has two bound states $\kappa_1 = M/\sqrt{2}$ and $\kappa_2 = \sqrt{2}M$. Inserting these values into (32) we obtain immediately

$$V_{\rm eff}^{\rm sub} = \left(\frac{1}{2\sqrt{6}} - \frac{3}{\sqrt{2}\pi}\right) M$$

in agreement with [11].

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